

# EFFECTIVE ACTION AND MEAN FERMION NUMBER DENSITY OF GRAPHENE IN CONSTANT MAGNETIC FIELD AT FINITE TEMPERATURE AND DENSITY

Alok Kumar<sup>1</sup>

Harish-Chandra Research Institute  
Chhatnag Road, Jhusi  
Allahabad 211 019  
Uttar Pradesh, India.

## *Abstract*

The effective action and the mean fermion number density of graphene in constant external magnetic field at finite temperature and density are calculated. Closed expressions for these are given and their variation with temperature are studied. It is found that the mean fermion number density peaks around a particular temperature, depending on the chemical potential at low temperatures. This feature is interpreted as 'condensation of fermions  $\bar{\psi}\psi$ ' in graphene. In future, it is interesting to extend and explore this calculation and the work of the reference [20] for the case of **Graphyne** [23].

Graphene is a honeycomb lattice of Carbon atoms with remarkable electronic properties [1,2]. Many of these properties are a low energy phenomena, governed by massless Dirac equation for the electrons in the tight-binding (hopping) model. This has been discovered in a seminal contribution of Semenoff [3] and exemplified by Mecklenberg and Regan [4] and Ando [5]. Thus, the excitations of graphene in the tight-binding approximation near the Fermi surface are described by (2+1)-dimensional massless Dirac equation, with velocities  $v_F \sim 10^6 \text{ ms}^{-1}$ . This description is accurate theoretically [3], [6], neglecting many body effects and experimentally [7,8], as the quasiparticles in graphene exhibit the linear dispersion relation  $E = pv_F$  (as if they are massless relativistic particles of speed  $v_F$ ) and by the observation of a relativistic analogue of integer quantum Hall effect. *Recently, it is shown mathematically in [20], that the linear dispersion relation (Dirac points) is generic in (non-relativistic) quantum crystals having the Honeycomb lattice symmetry.* It is important to realize that the quasi-particles of graphene have to be described by two-component wave functions, necessary to define relative contributions of sublattices [9]. Such a two-component description is similar to spinor wave functions in QED, with the 'spin index' for graphene indicating sublattices rather than the real spin of

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<sup>1</sup>e-mail: alok.academic@gmail.com [alok[POINT]academic[AT]gmail[POINT]com]

electrons.

A graphene layer has been extracted out of graphite [10] and such a discovery of 2-dimensional system attracted tremendous interest in theoretical physics, such as massless Dirac fermion, quantum Hall effect and even fractional quantum Hall effect [11], providing a framework to use the methods developed in quantum gauge field theories in graphene. Many of graphene physics is described by the Quantum Field Theory (QFT) [21]. *It is remarkable to note that many High Energy Physics (HEP) Experiments can be realised with the graphene table-top experiments. This statements is written in the spirit of Moore's Law for Electronics.* It is proposed in [22] how certain experimental results for carbon nanotubes (very closely related to graphene) could be re-interpreted in terms of the Kaluza-Klein mode of extra dimension. *Now the physics is more interesting with the advent of the new material Graphyne [23], which is more versatile than graphene due to its direction-dependent dirac cones.* A functional determinant approach with zeta function regularization has been employed by Beneventano and Santangelo [12] to study the quantum Hall effect in graphene and by Beneventano, Giacconi, Santangelo and Soldatic [13] to study the one-loop effects for massless Dirac fields coupled to electromagnetic backgrounds both at zero and finite temperature and density. The present author have studied  $SU(2)$  Yang-Mills theory at finite temperature in (3+1)-dimensions for understanding the vacuum in Quantum Chromo Dynamics [14] and so it is worthwhile to use the methods developed in [14] in graphene.

It is the purpose of this contribution to derive closed expression for the effective action  $\Gamma_{eff}$  for graphene (quasiparticles in graphene) in an external constant magnetic field at finite temperature and density. The 'mean fermionic number density'  $N_F = \frac{1}{\beta} \frac{\partial \Gamma_{eff}}{\partial \mu}$ , where  $\mu$  is the chemical potential, is calculated. We have used the  $\epsilon$ -regularization method of Salam and Strathdee [15] to regularize the divergences to obtain a finite action. The behaviour of the effective action and  $N_F$  with temperature has not been reported so far in the literature to the best of our knowledge.

Our results for  $\Gamma_{eff}$  and  $N_F$  are plotted against temperature for representative values of  $\mu$ . We find that the effective action peaks around a particular temperature for a given  $\mu$ . The 'mean fermionic number density' also peaks around the same temperature. This peaking of  $N_F$  is interpreted as a possible 'fermion pair condensation' or 'quasi-particle pair condensation' in graphene.

We consider massless electron in graphene coupled to an external magnetic field ( $U(1)$  gauge field) in (2+1)-dimensions, described by the Lagrangian density

$$\mathcal{L} = \bar{\psi} i \gamma^\mu D_\mu \psi; \quad \vec{A} = (-i \frac{\mu}{e}, 0, Bx), \quad (1)$$

where we use  $\hbar = v_F = 1$  units and the constant magnetic field is perpendicular to the plane of graphene.  $\mu$  is the chemical potential for the electrons. For the Euclidean version of (1) (useful to study the theory at finite temperature) we use

$$\begin{aligned}\gamma^0 &= -\sigma_3 ; \quad \gamma^1 = \sigma_2 ; \quad \gamma^2 = \sigma_1 ; \quad \{\gamma^i, \gamma^j\} = 2\delta^{ij}, \\ \mathcal{D} &= i\mathcal{D} - e\mathcal{A}.\end{aligned}\tag{2}$$

The eigenvalue equation

$$(i\mathcal{D} - e\mathcal{A})\psi = \lambda\psi,\tag{3}$$

gives

$$\lambda = \pm\sqrt{2NeB + (\omega_n + \mu)^2},\tag{4}$$

where the harmonic oscillator quantum number  $N = 0, 1, 2, \dots \infty$  and the Matsubara frequency  $\omega_n = (2n + 1)\frac{\pi}{\beta}$ ;  $n = -\infty$  to  $+\infty$ , with  $\beta = \frac{1}{kT}$ ,  $k$  being the Boltzmann constant. Equation (4) gives the Landau levels of the fermions in graphene. From the (Euclidean) partition function

$$Z = \int [d\psi][d\bar{\psi}] e^{-\int d^3x \mathcal{L}_E},$$

the effective action  $\Gamma_{eff} = \log Z = \log \det(i\mathcal{D} - e\mathcal{A})$  is given by

$$\Gamma_{eff} = -\frac{1}{4\beta} \left(\frac{eB}{2\pi}\right) \sum_{N=0}^{\infty} \sum_{n=-\infty}^{+\infty} \log\left(\frac{1}{\Lambda^2} \{2NeB + (\omega_n + \mu)^2\}\right),\tag{5}$$

where the prefactor  $\left(\frac{eB}{2\pi}\right)$  is the harmonic oscillator degeneracy factor and a dimensionful parameter  $\Lambda$  is introduced to render the argument of the logarithm dimensionless. In odd dimensions (such as here), there is another inequivalent representation for the gamma matrices, obtained for example, by reversing the sign of one or all the three gamma matrices in (2). Inclusion of this introduces a factor two for  $\Gamma_{eff}$ .

Now we use the  $\epsilon$ -regularization of Salam and Strathdee [15] (see also [16]) to write

$$\Gamma_{eff} = -\frac{1}{4\beta} \left(\frac{eB}{2\pi}\right) \sum_{N=0}^{\infty} \sum_{n=-\infty}^{\infty} \left(\frac{-i\epsilon}{\epsilon\Gamma(\epsilon)}\right) \int_0^{\infty} dt t^{-1+\epsilon} e^{-\frac{it}{\Lambda^2} (2NeB + (\omega_n + \mu)^2)}\tag{6}$$

in the limit  $\epsilon \rightarrow 0$ . After a Wick rotation  $t \rightarrow -i\tau$  and carrying out the  $N$ -sum, we find

$$\begin{aligned}\Gamma_{eff} &= -\frac{1}{4\beta} \left(\frac{eB}{2\pi}\right) \sum_{n=-\infty}^{\infty} \left(\frac{-1}{\epsilon\Gamma(\epsilon)}\right) \int_0^{\infty} \frac{d\tau}{(1 - e^{-2\tau eB/\Lambda^2})} \\ &\quad \tau^{-1+\epsilon} e^{-\frac{\tau}{\Lambda^2} (\omega_n + \mu)^2}.\end{aligned}\tag{7}$$

The  $n$ -sum can be done using Jacobi  $\theta_3$ -function to give

$$\Gamma_{eff} = -\frac{1}{4\beta} \left( \frac{eB}{2\pi} \right) \left( \frac{-1}{\epsilon\Gamma(\epsilon)} \right) \int_0^\infty \frac{d\tau}{(1 - e^{-2\tau eB/\Lambda^2})} \tau^{-1+\epsilon} e^{-\frac{\tau}{\Lambda^2} \left\{ \frac{\pi^2}{\beta^2} + \mu^2 + \frac{2\mu\pi}{\beta} \right\}} \theta_3 \left( \frac{2\pi i\tau}{\beta^2 \Lambda^2} + \frac{2\mu i\tau}{\beta \Lambda^2}, \frac{4\pi i\tau}{\beta^2 \Lambda^2} \right). \quad (8)$$

Using the property of  $\theta_3$ -function [17], namely

$$\theta_3(z, it) = t^{-\frac{1}{2}} e^{-\frac{\pi z^2}{t}} \theta_3\left(\frac{z}{it}, \frac{i}{t}\right), \quad (9)$$

we find

$$\Gamma_{eff} = \frac{\Lambda}{8\sqrt{\pi}} \left( \frac{eB}{2\pi} \right) \left( \frac{1}{\epsilon\Gamma(\epsilon)} \right) \int_0^\infty d\tau \frac{\tau^{-\frac{3}{2}+\epsilon}}{(1 - e^{-2\tau eB/\Lambda^2})} \theta_3\left(\frac{1}{2} + \frac{\mu\beta}{2\pi}, \frac{i\beta^2 \Lambda^2}{4\pi\tau}\right). \quad (10)$$

Rewriting  $\theta_3(z, t)$  as

$$\theta_3(z, t) = \sum_{\ell=-\infty}^{\infty} e^{i\pi t \ell^2} e^{2\pi i z \ell}, \quad (11)$$

equation (10) becomes

$$\Gamma_{eff} = \frac{\Lambda}{8\sqrt{\pi}} \left( \frac{eB}{2\pi} \right) \left( \frac{1}{\epsilon\Gamma(\epsilon)} \right) \int_0^\infty d\tau \frac{\tau^{-\frac{3}{2}+\epsilon}}{(1 - e^{-2\tau eB/\Lambda^2})} \left\{ 1 + 2 \sum_{\ell=1}^{\infty} e^{-\frac{\beta^2 \Lambda^2 \ell^2}{4\tau}} \cos(\ell\pi + \ell\mu\beta) \right\}. \quad (12)$$

In (12), the term without the  $\ell$ -sum has the integral over  $\tau$  is evaluated [18] and the limit  $\epsilon \rightarrow 0$  is taken to give

$$\Gamma_{eff}^{(1)} = \frac{1}{4} \left( \frac{eB}{2\pi} \right) \sqrt{2eB} \zeta\left(-\frac{1}{2}\right), \quad (13)$$

where the superscript denotes the contribution to  $\Gamma_{eff}$  coming from the term without the  $\ell$ -sum. This contribution corresponds to the effective action at zero temperature and zero chemical potential, and agrees with [13] and [19]. In (12), the term involving the  $\ell$ -sum is evaluated, first by writing

$$\frac{1}{(1 - e^{-2eB\tau/\Lambda^2})} = \sum_{n=0}^{\infty} e^{-\frac{2n eB}{\Lambda^2} \tau} = 1 + \sum_{n=1}^{\infty} e^{-\frac{2n eB}{\Lambda^2} \tau},$$

and then using the function  $K_\nu(xz)$  [18] and taking the limit  $\epsilon \rightarrow 0$ , to give

$$\begin{aligned}\Gamma_{eff}^{(2)} &= \frac{1}{4\pi} \left( \frac{eB}{2\pi} \right) \sum_{\ell=1}^{\infty} \cos(\ell\pi + \ell\mu\beta) \left\{ \frac{2}{\ell\beta} \Gamma\left(\frac{1}{2}\right) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \frac{2\sqrt{2}(2neB)^{\frac{1}{4}}}{\sqrt{\ell\beta}} K_{\frac{1}{2}}(\ell\beta\sqrt{2neB}) \right\},\end{aligned}\quad (14)$$

so that the effective becomes (adding (13) and (14))

$$\begin{aligned}\Gamma_{eff} &= \frac{1}{4} \left( \frac{eB}{2\pi} \right) \sqrt{2eB} \zeta\left(-\frac{1}{2}\right) \\ &\quad + \frac{1}{4\pi} \left( \frac{eB}{2\pi} \right) \sum_{\ell=1}^{\infty} \frac{\cos(\ell\pi + \ell\mu\beta)}{\beta\ell} \{2\sqrt{\pi} \\ &\quad + \sum_{n=1}^{\infty} 2\sqrt{2}\sqrt{\beta\ell\sqrt{2neB}} K_{\frac{1}{2}}(\beta\ell\sqrt{2neB})\}.\end{aligned}\quad (15)$$

Finally, using  $K_{\frac{1}{2}}(z) = \frac{\sqrt{\pi}}{\sqrt{2z}} e^{-z}$  [18], the effective action (15) is written as

$$\begin{aligned}\Gamma_{eff} &= \left( \frac{eB}{2\pi} \right) \left( \frac{1}{4} \sqrt{2eB} \zeta\left(-\frac{1}{2}\right) \right. \\ &\quad \left. + \frac{1}{2\sqrt{\pi}} \sum_{\ell=1}^{\infty} \frac{\cos(\ell\pi + \ell\mu\beta)}{\ell\beta} \left\{ 1 + \sum_{n=1}^{\infty} e^{-\beta\ell\sqrt{2neB}} \right\} \right).\end{aligned}\quad (16)$$

This is one of our main results, giving the effective energy density of the quasi-particles in graphene in a constant magnetic field at finite temperature and chemical potential. It is important to notice that the dimensionful parameter  $\Lambda$ , introduced in (5), has disappeared, unlike in (3+1)-dimensional gauge field theories [14,16]. The 'mean fermion number density' can be found using  $N_F = \frac{1}{\beta} \frac{\partial \Gamma_{eff}}{\partial \mu}$ .

In order to give the variation of  $\Gamma_{eff}$  with temperature, it is convenient to introduce

$$x = \beta\sqrt{2eB}; \quad y = \frac{\mu}{\sqrt{2eB}}; \quad \mu\beta = xy, \quad (17)$$

so that

$$\frac{\Gamma_{eff}}{\left( \frac{eB}{2\pi} \right) \sqrt{2eB}} = \frac{1}{4} \zeta\left(-\frac{1}{2}\right) + \frac{1}{2\sqrt{\pi}} \sum_{\ell=1}^{\infty} \frac{\cos(\ell\pi + \ell xy)}{\ell x} \left\{ 1 + \sum_{n=1}^{\infty} e^{-\ell\sqrt{nx}} \right\}, \quad (18)$$

and

$$\frac{N_F}{\left( \frac{eB}{2\pi} \right) \sqrt{2eB}} = -\frac{1}{2\sqrt{\pi}} \sum_{\ell=1}^{\infty} \frac{\sin(\ell\pi + \ell xy)}{x} \left\{ 1 + \sum_{n=1}^{\infty} e^{-\ell\sqrt{nx}} \right\},$$

$$\begin{aligned}
&= -\frac{1}{2\sqrt{\pi}} \left\{ -\frac{1}{2x} \tan\left(\frac{xy}{2}\right) \right. \\
&+ \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(\ell\pi + \ell xy)}{x} e^{-\ell\sqrt{n}x} \left. \right\}. \tag{19}
\end{aligned}$$

In Figure 1,  $\frac{\Gamma_{eff}}{\left(\frac{eB}{2\pi}\right)\sqrt{2eB}}$  is plotted against  $\frac{1}{x} = \frac{k}{\sqrt{2eB}} T$  for  $y = 0.5$  ( $\mu = 0.5\sqrt{2eB}$ ). The effective energy density peaks around  $\frac{1}{x} = 0.18$ . In Figure 2,  $\frac{|N_F|}{\left(\frac{eB}{2\pi}\right)\sqrt{2eB}}$  is shown against  $x^{-1}$  for  $y = 0.5$ . The 'mean fermion number density' peaks around  $x^{-1} = 0.18$ . In Figures 3 and 4, we give them for  $y = 0.8$ . We have found for  $y = 0.4, 0.5, 0.8, 1.0$ , the peaking occurs at different values of  $x^{-1}$ . The minimum value of  $y$  for the peak to occur is found to be  $y = 0.32$ .

This feature of peaking of  $N_F$  is interpreted as 'condensation of quasi-particle charge density'. Since the fermion number or charge density in this framework is proportional to  $\bar{\psi}\psi$ , it is the pair  $\bar{\psi}\psi$  (like Cooper pair) that condenses at low temperature and hence no violation of Pauli principle. This feature is different from the chiral symmetry breaking as it occurs around a particular temperature. Second, chiral symmetry breaking occurs when  $\frac{\partial\Gamma_{eff}}{\partial m}|_{m=0} \neq 0$ , where  $m$  is the fermion mass. Since the role of  $\mu$  is similar to that of  $m$ , we see from (18) and (19) that  $\frac{\partial\Gamma_{eff}}{\partial\mu}|_{\mu=0} = 0$  and so we realize a genuine condensation phenomenon. For the further deeper interpretation of the condensation we can look [24] and try to link our results in terms of material sciences.

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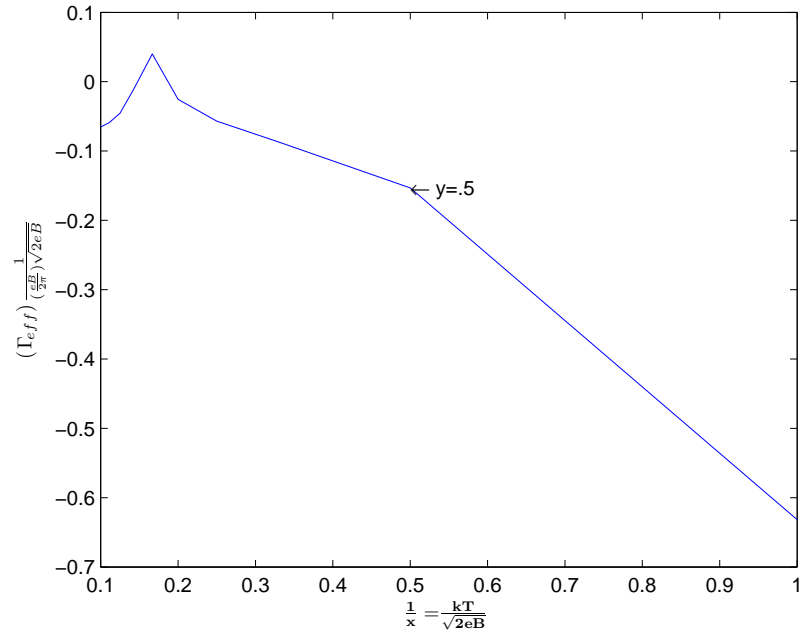


Figure 1: Variation of scaled effective energy density with scaled temperature for  $y=0.5$

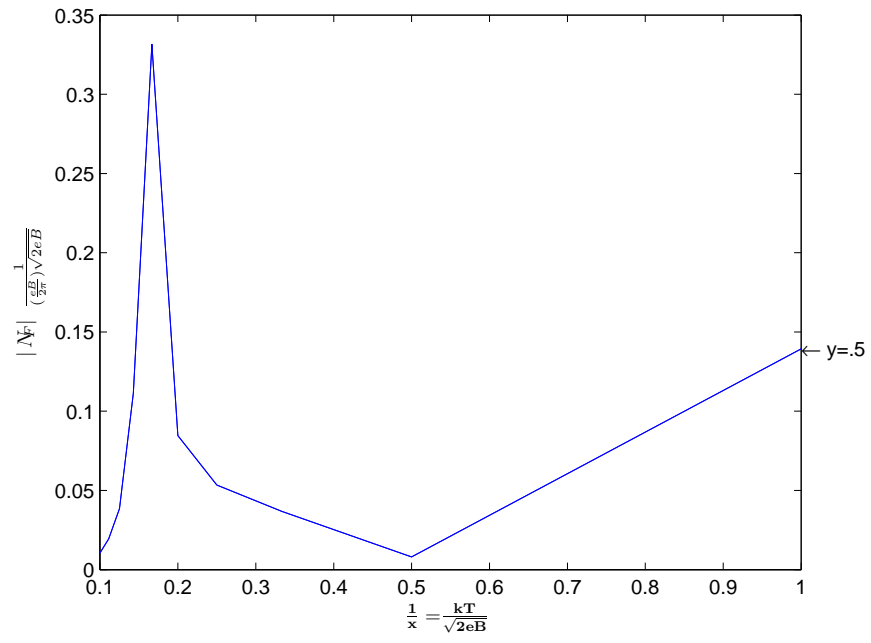


Figure 2: Variation of scaled charge density with scaled temperature for  $y=0.5$

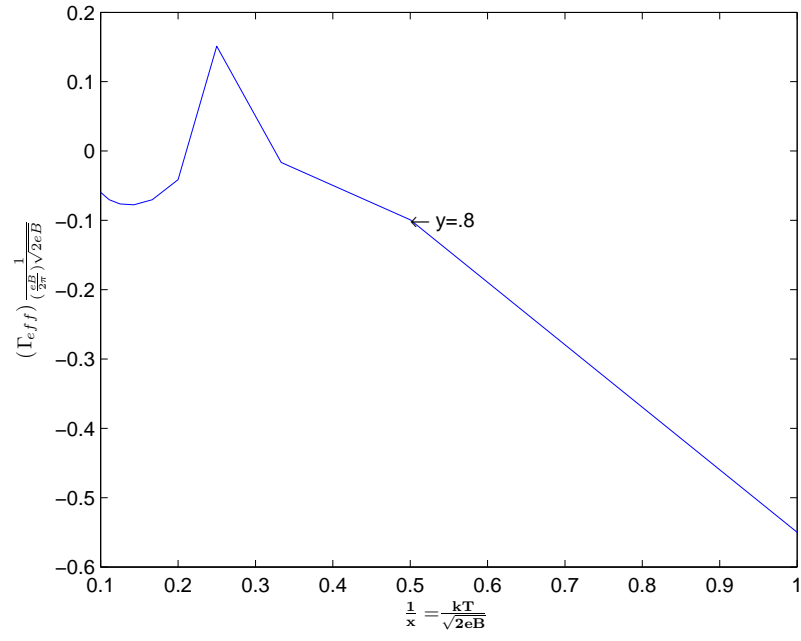


Figure 3: Variation of scaled effective energy density with scaled temperature for  $y=0.8$

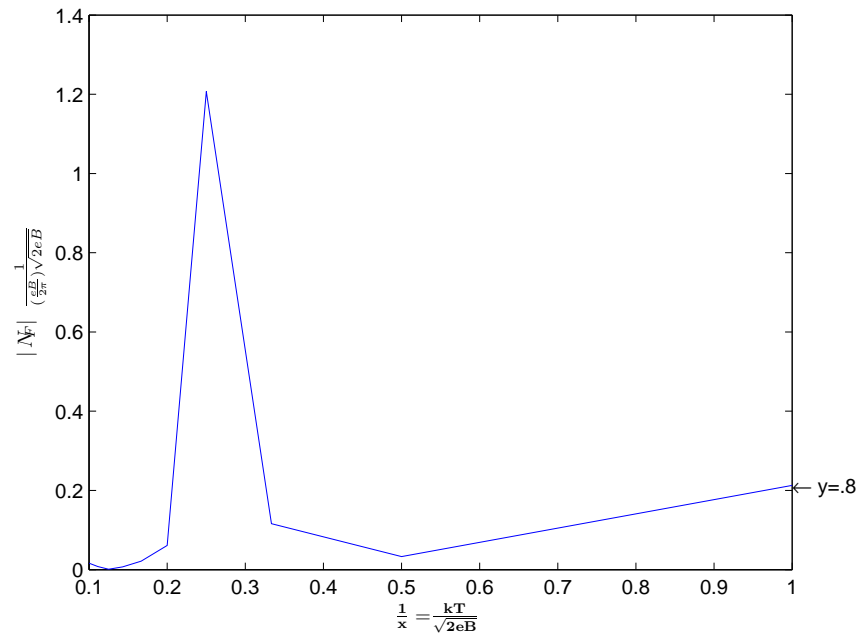


Figure 4: Variation of scaled charge density with scaled temperature for  $y=0.8$